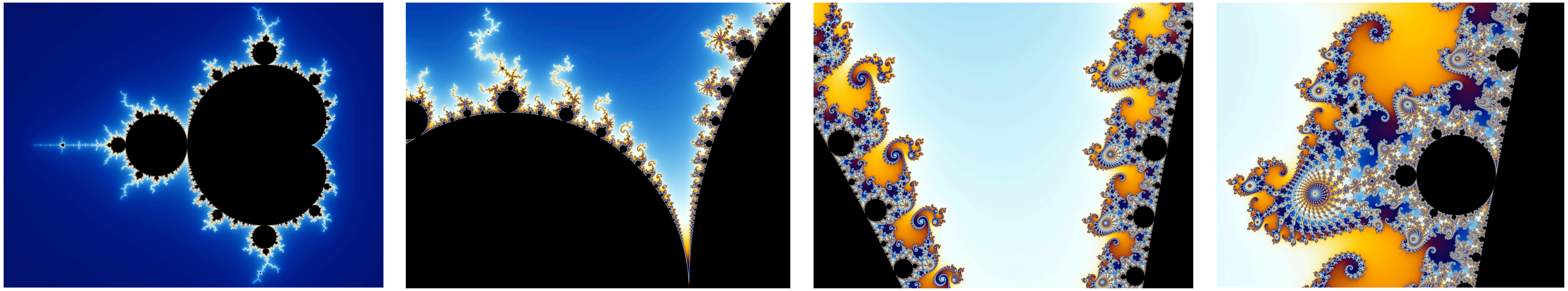
UNIVERSITY OF CAMBRIDGE

Fractals: between dimensions

Fractals are objects that exhibit complex structure at every scale. No matter how closely you zoom in on a fractal, its complexity doesn't diminish and you often see the same structures appearing again and again.



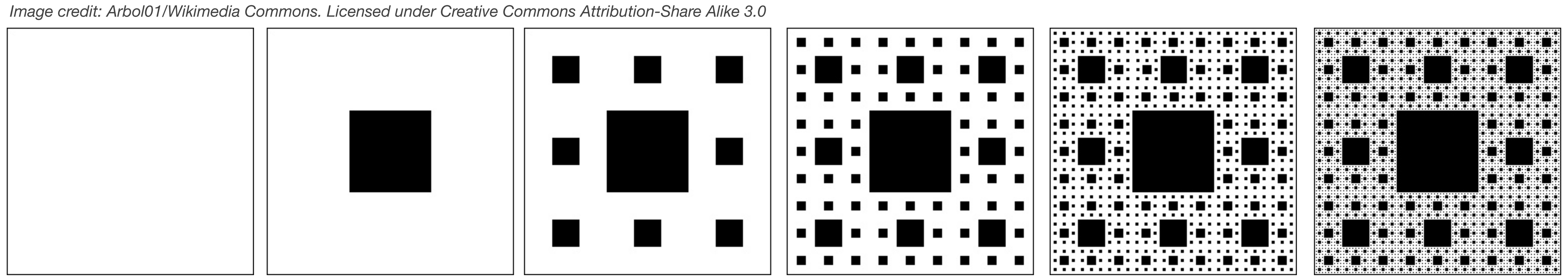
Left: The Mandelbrot set (far left) is one of the most famous fractals. The other three images show successive zooms to smaller and smaller scales, revealing the intricate complexity of the Mandelbrot set.

A famous example is the Sierpinski carpet. Start with a square and divide it into nine smaller squares. Remove the middle one of these nine squares. Now do the same for each of the eight remaining squares (dividing them into nine and removing the middle one) and repeat, ad infinitum. The shape you get in the end is riddled with holes. No matter how closely you zoom into it you won't find a single little patch that doesn't have any holes in it.

So how much area does Sierpinski's carpet occupy? The answer is zero. It has so many holes, there just isn't any area left. This means that it's fundamentally different from two-dimensional shapes, like squares or discs. If you take away the inside of a square, you are left with its sides, which are one-dimensional lines. So by analogy, perhaps we can think of Sierpinski's carpet as a one-dimensional shape too? The answer is again no. If you walked along the carpet's outline, you would find that it is infinitely long while confined to a finite area. Sierpinski's carpet is way more complex than an ordinary one-dimensional curve.

This is what defines fractals: they live in a strange world in between dimensions. Mathematicians have had to come up with new definitions of 'dimension' to sort out their place in the dimensional hierarchy.

Below: Images of Sierpinski carpet process.



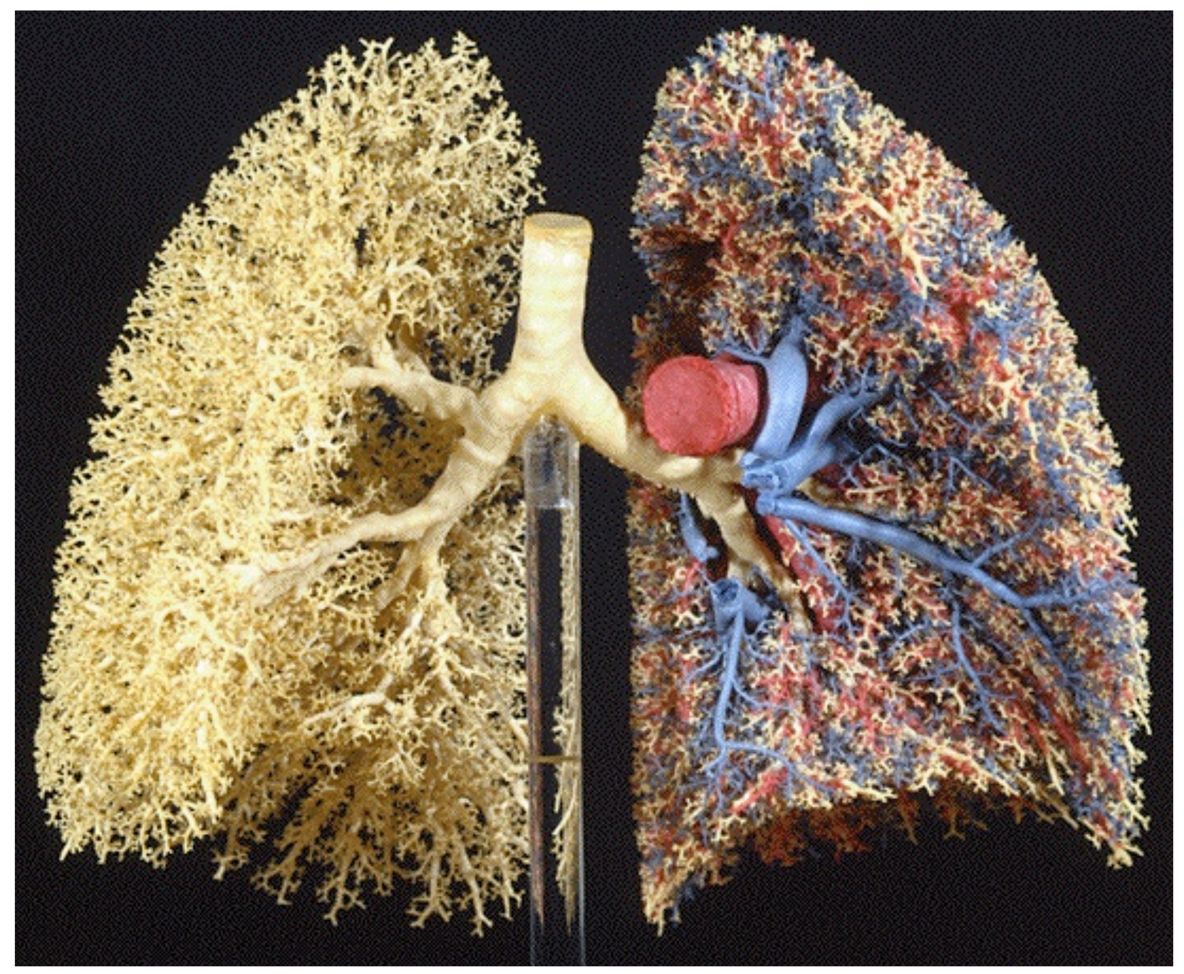
According to one of those definitions, Sierpinski's carpet has a dimension of around 1.9. Having a *fractional* dimension, one that's not a whole number, is what characterises fractals and what gave them their name.

Fractals may seem like mathematical oddities, but they are not. Many real-life shapes exhibit a self-similar complexity akin to those of fractals. A coastline looks just as crinkly whether you look at it on a satellite image or from a cliff overlooking a beach. Stock prices show a similar kind of variation over a period of a year as they do over a week. And our lungs exhibit an intricate branching structure similar to that of some fractals. That's no accident: lungs need to fit into the confined space of our chests (they need to have a small volume) but have as much surface area as possible in order to diffuse a maximal amount of oxygen through their surface. Being crinkly like a fractal makes that possible.

The Menger sponge is also a fractal. It's similar to the Sierpinski carpet, only that instead of starting with a square and removing little squares you start with a cube and remove little cubes. It has zero volume but its surface area is infinite. Its fractal dimension is around 2.7: it's more than two-dimensional but not quite three-dimensional.

Below: The intricate branching structure of our lungs, similar to fractals, guarantees maximal exposure to oxygen molecules.

Image credit: Image courtesy Professor Ewald R Weibel, Institute of Anatomy, University of Berne.



This poster is based on material written by Dr Marianne Freiberger, editor of Plus, the free online mathematics magazine produced by the Millennium Mathematics Project (maths.org) at the University of Cambridge.