Two and Four Dimensional Numbers

① (a) $\begin{pmatrix} x & -y \\ y & x \end{pmatrix} + \begin{pmatrix} u & -v \\ v & u \end{pmatrix} = \begin{pmatrix} u+x & -v-y \\ v+y & u+x \end{pmatrix}$

$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} u & -v \\ v & u \end{pmatrix} = \begin{pmatrix} ux-vy & -vx-uy \\ uy+vx & -vy+ux \end{pmatrix}$

$\begin{pmatrix} u & -v \\ v & u \end{pmatrix}\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} ux-vy & -uy-vx \\ vx+uy & -vy+ux \end{pmatrix}$

Are equal, so matrix multiplication in the set C^* is commutative

(b) Let $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$ so $A, B \in R^*$

$A+B = B+A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ 0 & a+b \end{pmatrix}$

$AB = BA = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} ab+0 & 0+0 \\ 0+0 & 0+ab \end{pmatrix} = \begin{pmatrix} ab & 0 \\ 0 & ab \end{pmatrix}$

A, B are also members of C^*, ∴ multiplication of them is commutative, as before

Addition and multiplication of matrices from the set R^* behaves almost identically to that of real numbers: the equivalent sum/product of real numbers makes up the top-left and bottom-left elements of the sum/product of the matrices

(c) Let $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $X = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ so $A, X \in C^*$

$$A+X = X+A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} a+x & -(b+y) \\ b+y & a+x \end{pmatrix}$$

$$AX = XA = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} ax-by & -(bx+ay) \\ bx+ay & ax-by \end{pmatrix}$$

(as before)

Compare to complex numbers:

Let $z_1 = a+bi$, $z_2 = x+yi$

$z_1 + z_2 = a+bi+x+yi = (a+x)+(b+y)i$

$z_1 z_2 = (a+bi)(x+yi) = ax+ayi+bxi-by = (ax-by)+(bx+ay)i$

$\therefore$ The sum/product of matrices ~~from~~ from C^* is equivalent to the sum/product of two complex numbers

Any matrix, Z, from C^* can be thought of as a representation of a complex number, such that, for some complex number z,

$$Z = \begin{pmatrix} Re(z) & -Im(z) \\ Im(z) & Re(z) \end{pmatrix}$$

(d) Let $z = 0+i$, Z be the matrix from C^* which represents z

From before, $Z = \begin{pmatrix} Re(z) & -Im(z) \\ Im(z) & Re(z) \end{pmatrix}$

$$\Rightarrow Z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Check: $Z^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0+(-1) & 0+0 \\ 0+0 & (-1)+0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$\therefore Z^2$ is equivalent to -1, so Z is equivalent to i

Two and Four Dimensional Numbers

② (a) $i^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -1+0 & 0+0 \\ 0+0 & 0+-1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$j^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0+(-1) & 0+0 \\ 0+0 & -1+0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$k^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0+(-1) & 0+0 \\ 0+0 & -1+0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is equivalent to the real number -1, so i, j and k are all equivalent to $\sqrt{-1}$

(b) $ij = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0+0 & i+0 \\ 0+i & 0+0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = k$

$ji = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0+0 & 0+(-i) \\ -i+0 & 0+0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -k$

$jk = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0+i & 0+0 \\ 0+0 & -i+0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i$

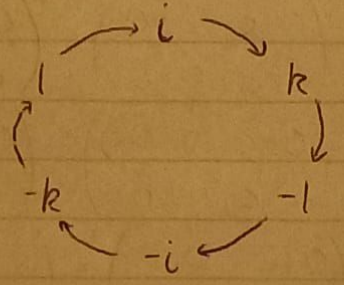
$kj = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0+(-i) & 0+0 \\ 0+0 & i+0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i$

$ki = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0+0 & 0+1 \\ -1+0 & 0+0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = j$

$ik = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0+0 & -1+0 \\ 0+1 & 0+0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -j$

(c) Terms in the sequence:

- $i = i$
- $ij = k$ (from before)
- $ijk = (k)k = k^2 = -1$
- $ijki = (-1)i = -i$
- $ijkij = (-i)j = -ij = -k$
- $ijkijk = (-k)k = -k^2 = -(-1) = 1$
- $ijkijki = (1)i = i$

To get from one term to the next, sub in the value from the previous term into the next term

The terms in the sequence follow this cycle (starting from i):